

Zeno's Paradoxes in the Mechanical World View

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There is a very reason to consider that to solve Zeno's paradoxes is to propose the theory of mechanical world view. We believe that this is not only our opinion but also most philosophers' opinion. Recently, in order to justify Heisenberg's uncertainty principle (cf. Rep. Math. Phys Vol. 29, No. 3, 1991) more firmly, we proposed the linguistic interpretation of quantum mechanics (called quantum and classical measurement theory), which was characterized as the metaphysical and linguistic turn of the Copenhagen interpretation. This turn from physics to language does not only extend quantum mechanics to classical systems but also yield the (quantum and classical) mechanical world view (and therefore, establish the method of science). If it be so, we may assert that Zeno's paradoxes (Flying Arrow Paradox, Achilles and the tortoise, etc.) were already solved in measurement theory. The purpose of this paper is to examine this assertion.

1. Introduction

1.1. Zeno's paradox (Achilles and the tortoise)

Although there are several Zeno's paradoxes (cf. [1]), we believe that they are essentially the same problem. Thus, in this paper, we devote ourselves to the most famous Zeno's paradox (i.e., Achilles and the tortoise).

(A) [Achilles and the tortoise] In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. Is it true?

1.2. The formulation

In what follows we shall introduce well known idea about the above (A). For example, assume that the velocity v_q [resp. v_s] of the quickest [resp. slowest] runner is equal to $v(> 0)$ [resp. γv ($0 < \gamma < 1$)]. And further, assume that the position of the quickest [resp. slowest] runner at time $t = 0$ is equal to 0 [resp. a (> 0)]. Thus, we can assume that the position $\xi(t)$ of the quickest runner and the position $\eta(t)$ of the slowest runner at time t (≥ 0) is respectively represented by

$$\begin{cases} \xi(t) = vt \\ \eta(t) = \gamma vt + a \end{cases} \quad (1)$$

1.3. Calculations

The formula (1) can be calculated as follows (i.e., (i) or (ii)):

[(i): Algebraic calculation of (1)]:

Solving $\xi(s_0) = \eta(s_0)$, that is,

$$vs_0 = \gamma vs_0 + a$$

we get $s_0 = \frac{a}{(1-\gamma)v}$. That is, at time $s_0 = \frac{a}{(1-\gamma)v}$, the fast runner catches up with the slow runner.

[(ii): Iterative calculation of (1)]:

Define t_k ($k = 0, 1, \dots$) such that, $t_0 = 0$ and

$$t_{k+1} = \gamma vt_k + a \quad (k = 0, 1, 2, \dots)$$

Thus, we see that $t_k = \frac{(1-\gamma^k)a}{(1-\gamma)v}$ ($k = 0, 1, \dots$). Then, we have that

$$\begin{aligned} (\xi(t_k), \eta(t_k)) &= \left(\frac{(1-\gamma^k)a}{1-\gamma}, \frac{(1-\gamma^{k+1})a}{1-\gamma} \right) \\ &\rightarrow \left(\frac{a}{1-\gamma}, \frac{a}{1-\gamma} \right) \end{aligned} \quad (2)$$

as $k \rightarrow \infty$. Therefore, the quickest runner catches up with the slowest at time $s_0 = \frac{a}{(1-\gamma)v}$.

[(iii): Conclusion]: After all, by the above (i) or (ii), we can conclude that

(B) the quickest runner can overtake the slowest at time $s_0 = \frac{a}{(1-\gamma)v}$.

1.4. What is Zeno's paradox?

It is a matter of course that what is important is the formulation (in Section 1.2) and not the calculation (in Section 1.3). That is, we believe that what Zeno (or, the philosophers who inherit Zeno) wanted to ask is as follows:

- (C) Why do we believe in the formula (1)? Or, what kind of world view is the (1) based on?

The purpose of this paper is to answer this question (C), that is, to assert that

- (D) the formula (1) is based on measurement theory.

There is a very reason to consider that the world view in (C) is called the mechanical world view. Thus, if the assertion (D) is generally accepted, we may assert that measurement theory is just the theory of mechanical world view.

2. Measurement Theory

2.1. Overview: Measurement theory

In this section, we shall mention the overview of measurement theory (or in short, MT).

It is well known (*cf.* [2]) that quantum mechanics is formulated in an operator algebra $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H=1} \|Fu\|_H$) as follows:

- (E) quantum mechanics
(physics)

$$= \underset{\text{(probabilistic interpretation)}}{[\text{quantum measurement}]} + \underset{\text{(kinetic equation)}}{[\text{causality}]}$$

Also, the Copenhagen interpretation due to N. Bohr (et al.) is characterized as the guide to the usage of quantum mechanics (E). Although quantum mechanics (E) with the Copenhagen interpretation is generally accepted as one of the most trustworthy theories in science, it should be noted that there is no definitive statement of the Copenhagen interpretation, that is, there are a number of ideas that are associated with the Copenhagen interpretation. We do not think that this fact is desirable.

Measurement theory (mentioned in the following sections or refs. [3–13]) is, by an analogy of the (E), constructed as the mathematical theory formulated in a certain C^* -algebra \mathcal{A} (i.e., a norm closed subalgebra in $B(H)$, *cf.* [14]) as follows:

- (F) measurement theory
(language)

$$= \underset{\text{(Axiom 1 in Section 2.2)}}{[\text{measurement}]} + \underset{\text{(Axiom 2 in Section 2.2)}}{[\text{causality}]}$$

Note that this theory (F) is not physics but a kind of language based on “the mechanical world view” since it is a mathematical generalization of quantum mechanics (E).

When $\mathcal{A} = B_c(H)$, the C^* -algebra composed of all compact operators on a Hilbert space H , the (F) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when \mathcal{A} is commutative (that is, when \mathcal{A} is characterized by $C_0(\Omega)$, the C^* -algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (*cf.* [14, 15])), the (F) is called classical measurement theory. Thus, we have the following classification:

- (G) measurement theory

$$= \begin{cases} \text{quantum measurement theory} \\ \quad \text{(when } \mathcal{A} = B_c(H) \text{)} \\ \text{classical measurement theory} \\ \quad \text{(when } \mathcal{A} = C_0(\Omega) \text{)} \end{cases}$$

That is, this theory covers several conventional system theories (i.e., statistics, dynamical system theory, quantum system theory).

2.2. Measurement theory

Measurement theory (F) has two formulations (i.e., the C^* -algebraic formulation and the W^* -algebraic formulation, *cf.* [6]). In this paper, we devote ourselves to the W^* -algebraic formulation. of the measurement theory (F).

Let $\mathcal{A}(\subseteq B(H))$ be a C^* -algebra, and let \mathcal{A}^* be the dual Banach space of \mathcal{A} . That is, $\mathcal{A}^* = \{\rho \mid \rho \text{ is a continuous linear functional on } \mathcal{A}\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid F \in \mathcal{A} \text{ such that } \|F\|_{\mathcal{A}} (= \|F\|_{B(H)}) \leq 1\}$. Define the *mixed state* ρ ($\in \mathcal{A}^*$) such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \geq 0$ for all $F \in \mathcal{A}$ such that $F \geq 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*) = \{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.$$

A mixed state $\rho(\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that “ $\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$ ” implies “ $\rho = \rho_1 = \rho_2$ ”. Put

$$\mathfrak{S}^p(\mathcal{A}^*) = \{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\},$$

which is called a *state space*. It is well known (cf. [14]) that $\mathfrak{S}^p(B_c(H)^*) = \{|u\rangle\langle u| \text{ (i.e., the Dirac notation)} \mid \|u\|_H = 1\}$, and $\mathfrak{S}^p(C_0(\Omega)^*) = \{\delta_{\omega_0} \mid \delta_{\omega_0} \text{ is a point measure at } \omega_0 \in \Omega\}$, where $\int_{\Omega} f(\omega) \delta_{\omega_0}(d\omega) = f(\omega_0)$ ($\forall f \in C_0(\Omega)$). The latter implies that $\mathfrak{S}^p(C_0(\Omega)^*)$ can be also identified with Ω (called a *spectrum space* or simply *spectrum*) such as

$$\mathfrak{S}^p(C_0(\Omega)^*) \ni \delta_{\omega} \leftrightarrow \omega \in \underset{\text{(state space)}}{\Omega} \underset{\text{(spectrum)}}{\quad} \quad (3)$$

Consider the pair $[\mathcal{A}, \mathcal{N}]_{B(H)}$, called a *basic structure*. Here, $\mathcal{A}(\subseteq B(H))$ is a C^* -algebra, and $\mathcal{N}(\mathcal{A} \subseteq \mathcal{N} \subseteq B(H))$ is a particular C^* -algebra (called a W^* -algebra) such that \mathcal{N} is the weak closure of \mathcal{A} in $B(H)$. Let \mathcal{N}_* be the pre-dual Banach space.

For example, we see (cf. [14]) that, when $\mathcal{A} = B_c(H)$,

- (i) \mathcal{A}^* = “trace class”, $\mathcal{N} = B(H)$, \mathcal{N}_* = “trace class”.

Also, when $\mathcal{A} = C_0(\Omega)$,

- (ii) \mathcal{A}^* = “the space of all signed measures on Ω ”, $\mathcal{N} = L^\infty(\Omega, \nu)(\subseteq B(L^2(\Omega, \nu)))$, $\mathcal{N}_* = L^1(\Omega, \nu)$, where ν is some measure on Ω (cf. [14]).

In this paper, $L^\infty(\Omega, \nu)$ and $L^1(\Omega, \nu)$ is often written by $L^\infty(\Omega)$ and $L^1(\Omega)$ respectively.

For instance, in the above (ii) we must clarify the meaning of the “value” of $F(\omega_0)$ for $F \in L^\infty(\Omega, \nu)$ and $\omega_0 \in \Omega$. An element $F(\in \mathcal{N})$ is said to be *essentially continuous at* $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$, if there uniquely exists a complex number α such that

- (H) if $\rho(\in \mathcal{N}_*, \|\rho\|_{\mathcal{N}_*} = 1)$ converges to $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$ in the sense of weak* topology of \mathcal{A}^* , that is,

$$\rho(G) \longrightarrow \rho_0(G) \quad (\forall G \in \mathcal{A}(\subseteq \mathcal{N})), \quad (4)$$

then $\rho(F)$ converges to α .

And the value of $\rho_0(F)$ is defined by the α .

According to the noted idea [16], an *observable* $\mathbf{O} := (X, \mathcal{F}, F)$ in \mathcal{N} is defined as follows:

- (i) [σ -field] X is a set, $\mathcal{F}(\subseteq \mathcal{P}(X))$, the power set of X is a σ -field of X , that is, “ $\Xi_1, \Xi_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^\infty \Xi_n \in \mathcal{F}$ ”, “ $\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$ ”.
- (ii) [Countable additivity] F is a mapping from \mathcal{F} to \mathcal{N} satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in \mathcal{N} such that $0 \leq F(\Xi)$

$\leq I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in \mathcal{N} respectively. (c): for any countable decomposition $\{\Xi_1, \Xi_2, \dots, \Xi_n, \dots\}$ of Ξ (i.e., $\Xi, \Xi_n \in \mathcal{F}$ ($n = 1, 2, 3, \dots$), $\bigcup_{n=1}^\infty \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ ($i \neq j$)), it holds that $F(\Xi) = \sum_{n=1}^\infty F(\Xi_n)$ in the sense of weak* topology in \mathcal{N} .

With any *system* S , a basic structure $[\mathcal{A}, \mathcal{N}]_{B(H)}$ can be associated in which the measurement theory (F) of that system can be formulated. A *state* of the system S is represented by an element $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$ and an *observable* is represented by an observable $\mathbf{O} := (X, \mathcal{F}, F)$ in \mathcal{N} . Also, the *measurement of the observable* \mathbf{O} *for the system* S *with the state* ρ is denoted by $M_{\mathcal{N}}(\mathbf{O}, S_{[\rho]})$ (or more precisely, $M_{\mathcal{N}}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\rho]})$). An observer can obtain a measured value $x(\in X)$ by the measurement $M_{\mathcal{N}}(\mathbf{O}, S_{[\rho]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born’s probabilistic interpretation of quantum mechanics (E). And thus, it is a statement without reality.

Now we can present Axiom 1 in the W^* -algebraic formulation as follows.

Axiom 1 [Measurement]. *The probability that a measured value $x(\in X)$ obtained by the measurement $M_{\mathcal{N}}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\rho_0]})$ belongs to a set $\Xi(\in \mathcal{F})$ is given by $\rho_0(F(\Xi))$ if $F(\Xi)$ is essentially continuous at $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$.*

Next, we explain Axiom 2. Let $[\mathcal{A}_1, \mathcal{N}_1]_{B(H_1)}$ and $[\mathcal{A}_2, \mathcal{N}_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2} : \mathcal{N}_2$ (with weak* topology) $\rightarrow \mathcal{N}_1$ (with weak* topology) is called a *Markov operator*, if it satisfies that (i): $\Phi_{1,2}(F_2) \geq 0$ for any non-negative element F_2 in \mathcal{N}_2 , (ii): $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in \mathcal{N}_k , ($k = 1, 2$). In addition to the above (i) and (ii), in this paper we assume that $\Phi_{1,2}(\mathcal{A}_2) \subseteq \mathcal{A}_1$ and $\sup\{\|\Phi_{1,2}(F_2)\|_{\mathcal{A}_1} \mid F_2 \in \mathcal{A}_2 \text{ such that } \|F_2\|_{\mathcal{A}_2} \leq 1\} = 1$.

It is clear that the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \rightarrow \mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*(\mathfrak{S}^m(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$.

Here note that, for any observable $\mathbf{O}_2 := (X, \mathcal{F}, F_2)$ in \mathcal{N}_2 , the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in \mathcal{N}_1 .

Let (T, \leq) be a tree, i.e., a partial ordered set such that “ $t_1 \leq t_3$ and $t_2 \leq t_3$ ” implies “ $t_1 \leq t_2$ or $t_2 \leq t_1$ ”. Put $T_{\leq}^2 = \{(t_1, t_2) \in T^2 \mid t_1 \leq t_2\}$. Here, note that T is not necessarily finite.

Assume the completeness of the ordered set T . That is, for any subset $T'(\subseteq T)$ bounded from below

(i.e., there exists $t' \in T$ such that $t' \leq t$ ($\forall t \in T'$)), there uniquely exists an element $\inf(T') \in T$ satisfying the following conditions, (i): $\inf(T') \leq t$ ($\forall t \in T'$), (ii): if $s \leq t$ ($\forall t \in T'$), then $s \leq \inf(T')$.

The family $\{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}$ is called a *Markov relation* (due to the Heisenberg picture), if it satisfies the following conditions (i) and (ii).

- (i) With each $t \in T$, a basic structure $[\mathcal{A}_t, \mathcal{N}_t]_{B(H_t)}$ is associated.
- (ii) For every $(t_1, t_2) \in T_{\leq}^2$, a Markov operator $\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}$ is defined. And it satisfies that $\Phi_{t_1, t_2} \Phi_{t_2, t_3} = \Phi_{t_1, t_3}$ holds for any $(t_1, t_2), (t_2, t_3) \in T_{\leq}^2$.

When $\Phi_{t_1, t_2}^* (\mathfrak{S}^p(\mathcal{A}_{t_1}^*)) \subseteq (\mathfrak{S}^p(\mathcal{A}_{t_2}^*))$ holds for any $(t_1, t_2) \in T_{\leq}^2$, the Markov relation is said to be deterministic. Note that the classical deterministic Markov relation is represented by $\{\phi_{t_1, t_2} : \Omega_{t_1} \rightarrow \Omega_{t_2}\}_{(t_1, t_2) \in T_{\leq}^2}$, where the continuous map $\phi_{t_1, t_2} : \Omega_{t_1} \rightarrow \Omega_{t_2}$ is defined by

$$\Phi_{t_1, t_2}^*(\delta_{\omega_1}) = \delta_{\phi_{t_1, t_2}(\omega_1)} \quad (\forall \omega_1 \in \Omega_1)$$

Now Axiom 2 is presented as follows:

Axiom 2 [Causality]. *The causality is represented by a Markov relation $\{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}$.*

2.3. The Linguistic Interpretation

According to [8,9,11], we shall explain the linguistic interpretation of quantum mechanics. The measurement theory (F) asserts

- (I) Obey Axioms 1 and 2. And, describe any ordinary phenomenon according to Axioms 1 and 2 (in spite that Axioms 1 and 2 can not be tested experimentally).

Still, most readers may be perplexed how to use Axioms 1 and 2 since there are various usages. Thus, the following problem is significant.

- (J) How should Axioms 1 and 2 be used?

Note that reality is not reliable since Axioms 1 and 2 are statements without reality.

Here, in spite of the linguistic turn (Figure 1:③) and the mathematical generalization from $B(H)$ to a C^* -algebra \mathcal{A} , we consider that the dualism (i.e., the main spirit of so called Copenhagen interpretation) of quantum mechanics is inherited to measurement theory. Thus, we present the following interpretation (K) [= (K₁)–(K₃)]. That is, as the answer to the question (J), we propose:

- (K₁) Consider the dualism composed of “observer”

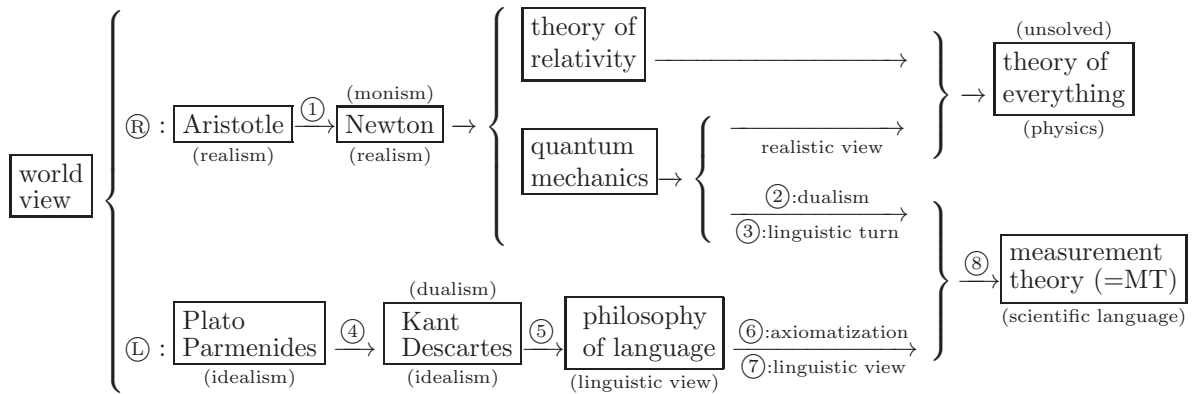


Figure 1. The development of the world views. For the explanations of ①–⑧, see [9, 11].

and “system(=measuring object)”. And therefore, “observer” and “system” must be absolutely separated. See **Figure 2**.

(K₂) Only one measurement is permitted. And thus, the state after a measurement is meaningless since it can not be measured any longer. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted rather than the Schrödinger picture.

(K₃) Also, the observer does not have the space-time. Thus, the question: “When and where is a measured value obtained?” is out of measurement theory,

and so on.

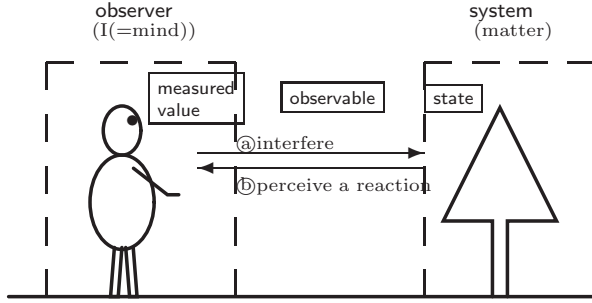


Figure 2. Dualism in MT.

In this sense, we consider that measurement theory holds as a kind of language-game (with the rule (Axioms 1 and 2, Interpretation (K)), and therefore, measurement theory is regarded as the axiomatization (**Figure 1:⑥**) of the philosophy of language (i.e., Saussure’s linguistic world view). For the precise explanations of **Figures 1 and 2**, see [9, 11].

Note that quantum mechanics (E) has many interpretations. On the other hand, we believe that the interpretation of measurement theory (F) is uniquely determined as in the above. This is our main reason to propose the linguistic interpretation of quantum mechanics. We believe that this uniqueness is essential to the justification of Heisenberg’s uncertainty principle (*cf.* [11, 17]).

Example 1 [Simultaneous measurement] For each $k = 1, 2, \dots, K$, consider a measurement $M_{\mathcal{N}}(O_k := (X_k, \mathcal{F}_k, F_k), S_{[\rho]})$. However, since the (K₂) says that only one measurement is permitted, the $\{M_{\mathcal{N}}(O_k, S_{[\rho]})\}_{k=1}^K$ should be reconsidered in what

follows. Under the commutativity condition such that

$$F_i(\Xi_i)F_j(\Xi_j) = F_j(\Xi_j)F_i(\Xi_i) \quad (5)$$

$$(\forall \Xi_i \in \mathcal{F}_i, \forall \Xi_j \in \mathcal{F}_j, i \neq j),$$

we can define the simultaneous observable $\times_{k=1}^K O_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \times_{k=1}^K F_k)$ in \mathcal{N} such that

$$(\times_{k=1}^K F_k)(\times_{k=1}^K \Xi_k) = F_1(\Xi_1)F_2(\Xi_2) \cdots F_K(\Xi_K) \quad (6)$$

$$(\forall \Xi_k \in \mathcal{F}_k, \forall k = 1, \dots, K).$$

where $(\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k)$ is the product measurable space. Then, the above $\{M_{\mathcal{N}}(O_k, S_{[\rho]})\}_{k=1}^K$ is, under the commutativity condition (5), represented by the simultaneous measurement $M_{\mathcal{N}}(\times_{k=1}^K O_k, S_{[\rho]})$.

Example 2 [How to use Axiom 2 (Causality)] Consider a finite tree $(T := \{t_0, t_1, \dots, t_n\}, \leq)$ with the root t_0 . This is also characterized by the map $\pi : T \setminus \{t_0\} \rightarrow T$ such that $\pi(t) = \max\{s \in T \mid s < t\}$. Let $\{\Phi_{t',t} : \mathcal{N}_t \rightarrow \mathcal{N}_{t'}\}_{(t',t) \in T_{\leq}^2}$ be a Markov relation, which is also represented by $\{\Phi_{\pi(t),t} : \mathcal{N}_t \rightarrow \mathcal{N}_{\pi(t)}\}_{t \in T \setminus \{t_0\}}$. Let an observable $O_t := (X_t, \mathcal{F}_t, F_t)$ in the \mathcal{N}_t be given for each $t \in T$. Consider the pair $\mathbb{O}_T = [\{O_t\}_{t \in T}, \{\Phi_{\pi(t),t} : \mathcal{N}_t \rightarrow \mathcal{N}_{\pi(t)}\}_{t \in T \setminus \{t_0\}}]$, which is called a sequential observable. Let $\rho_0 \in \mathfrak{S}^p(\mathcal{A}_{t_0}^*)$. Consider “measurement” such as

(L) a measurement of a sequential observable \mathbb{O}_T for the system with a ρ_0 .

where the meaning of the (L) is not clear yet. Recalling that the (K₃) says that a state never moves, we consider the meaning of the (L) as follows: For each $s \in T$, put $T_s = \{t \in T \mid t \geq s\}$. And define the observable $\hat{O}_s = (\times_{t \in T_s} X_t, \boxtimes_{t \in T_s} \mathcal{F}_t, \hat{F}_s)$ in \mathcal{N}_s (due to Heisenberg picture) as follows:

$$\hat{O}_s = \begin{cases} O_s & (\text{if } s \in T \setminus \pi(T)) \\ O_s \times (\times_{t \in \pi^{-1}(\{s\})} \Phi_{\pi(t),t} \hat{O}_t) & (\text{if } s \in \pi(T)) \end{cases} \quad (7)$$

if the commutativity condition holds (i.e., if the simultaneous observable $O_s \times (\times_{t \in \pi^{-1}(\{s\})} \Phi_{\pi(t),t} \hat{O}_t)$ exists) for each $s \in \pi(T)$. Using (7) iteratively, we can finally obtain the observable \hat{O}_{t_0} (which is also denoted by \hat{O}_T) in \mathcal{N}_{t_0} , which is regarded as the realization of \mathbb{O}_T . Thus the above (L) is represented by the measurement $M_{\mathcal{N}_{t_0}}(\hat{O}_T, S_{[\rho_0]})$.

2.4. Leibniz-Clarke Correspondence (Space-Time Problem)

Consider a basic structure $[\mathcal{A}, \mathcal{N}]_{B(H)}$. Let $\mathcal{A}_S (\subseteq \mathcal{N})$ be the commutative C^* -subalgebra. Note that \mathcal{A}_S is represented such that $\mathcal{A}_S = C_0(\Omega_S)$ for some locally compact Hausdorff space Ω_S (cf. [14]). As seen in the formula (3), the Ω_S is called a *spectrum*. For example, consider one particle quantum system, formulated in a basic structure:

$$[B_c(L^2(\mathbb{R}^3)), B(L^2(\mathbb{R}^3))]_{B(L^2(\mathbb{R}^3))}.$$

Then, we can choose the commutative C^* -algebra $C_0(\mathbb{R}^3) (\subset B(L^2(\mathbb{R}^3)))$, and thus, we get the spectrum \mathbb{R}^3 . This simple example will make us propose the (M₂) later.

In Leibniz-Clarke correspondence (1715–1716), they (i.e., Leibniz and Clarke(=Newton’s friend)) discussed “space-time problem”. Their ideas are summarized as follows:

$$(M_1) \left\{ \begin{array}{ll} \textcircled{R}: \text{Newton, Clarke} \cdots & \begin{array}{c} \text{(space-time in physics)} \\ \text{realistic space-time} \end{array} \\ \text{(realistic world view)} & \\ \textcircled{L}: \text{Leibniz} \cdots & \begin{array}{c} \text{(space-time in language)} \\ \text{linguistic space-time} \end{array} \\ \text{(linguistic world view)} & \end{array} \right.$$

That is, Newton considered “What is space-time?”. On the other hand, Leibniz considered “How should space-time be represented?”, though he did not propose his language. Measurement theory is in Leibniz’s side, and asserts that

(M₂) Space should be described as a kind of spectrum.

And time should be described as a kind of tree.

In other words, time is represented by a parameter t in a linear ordered tree T .

Therefore, we think that the Leibniz-Clarke debates should be essentially regarded as “the linguistic world view \textcircled{L} ” vs. “the realistic world view \textcircled{R} ” in **Figure 1**. Hence, the statement (M₂) should be added to Interpretation (K) as sub-interpretation of measurement theory.

3. Preliminary mathematical results

Let $\hat{\Lambda}$ be an index set. For each $\lambda \in \hat{\Lambda}$, consider a set X_λ . For any subsets $\Lambda_1 \subseteq \Lambda_2 (\subseteq \hat{\Lambda})$, P_{Λ_1, Λ_2} is defined by the natural projection such that:

$$\times_{\lambda \in \Lambda_2} X_\lambda \ni (x_\lambda)_{\lambda \in \Lambda_2} \xrightarrow{P_{\Lambda_1, \Lambda_2}} (x_\lambda)_{\lambda \in \Lambda_1} \in \times_{\lambda \in \Lambda_1} X_\lambda.$$

For each $\lambda \in \hat{\Lambda}$, consider an observable $(X_\lambda, \mathcal{F}_\lambda, F_\lambda)$ in W^* -algebra \mathcal{N} . And consider the quasi-product observable $\mathbf{O} \equiv (\times_{\lambda \in \hat{\Lambda}} X_\lambda, \boxtimes_{\lambda \in \hat{\Lambda}} \mathcal{F}_\lambda, F_{\hat{\Lambda}})$ of $\{ (X_\lambda, \mathcal{F}_\lambda, F_\lambda) \mid \lambda \in \hat{\Lambda} \}$, which is defined by the observable such that:

$$\hat{F}_{\hat{\Lambda}}(P_{\{\lambda\}, \hat{\Lambda}}^{-1}(\Xi_\lambda)) = F_\lambda(\Xi_\lambda) \quad (\forall \Xi_\lambda \in \mathcal{F}_\lambda, \forall \lambda \in \hat{\Lambda}), \quad (8)$$

though the existence and the uniqueness of a quasi-product observable are not guaranteed in general. The following theorem says something about the existence and uniqueness of the quasi-product observable.

Theorem 1 [W^* -algebraic Kolmogorov extension theorem, cf. [6, 18]]. *For each $\lambda \in \hat{\Lambda}$, consider a Borel measurable space $(X_\lambda, \mathcal{F}_\lambda)$, where X_λ is a separable complete metric space. Define the set $\mathcal{P}_0(\hat{\Lambda})$ such as $\mathcal{P}_0(\hat{\Lambda}) \equiv \{ \Lambda \subseteq \hat{\Lambda} \mid \Lambda \text{ is finite} \}$. Assume that the family of the observables $\{ \mathbf{O}_\Lambda \equiv (\times_{\lambda \in \Lambda} X_\lambda, \boxtimes_{\lambda \in \Lambda} \mathcal{F}_\lambda, F_\Lambda) \mid \Lambda \in \mathcal{P}_0(\hat{\Lambda}) \}$ in \mathcal{N} satisfies the following “consistency condition”:*

(N) *for any $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\hat{\Lambda})$ such that $\Lambda_1 \subseteq \Lambda_2$, it holds that*

$$F_{\Lambda_2}(P_{\Lambda_1, \Lambda_2}^{-1}(\Xi_{\Lambda_1})) = F_{\Lambda_1}(\Xi_{\Lambda_1}) \quad (\forall \Xi_{\Lambda_1} \in \boxtimes_{\lambda \in \Lambda_1} \mathcal{F}_\lambda). \quad (9)$$

Then, there uniquely exists the observable $\hat{\mathbf{O}}_{\hat{\Lambda}} \equiv (\times_{\lambda \in \hat{\Lambda}} X_\lambda, \boxtimes_{\lambda \in \hat{\Lambda}} \mathcal{F}_\lambda, F_{\hat{\Lambda}})$ in \mathcal{N} such that:

$$\begin{aligned} \hat{F}_{\hat{\Lambda}}(P_{\Lambda, \hat{\Lambda}}^{-1}(\Xi_\Lambda)) &= F_\Lambda(\Xi_\Lambda) \\ (\forall \Xi_\Lambda \in \boxtimes_{\lambda \in \Lambda} \mathcal{F}_\lambda, \forall \Lambda \in \mathcal{P}_0(\hat{\Lambda})). \end{aligned}$$

Proof. See [6]. □

As mentioned in [8], we believe that

(O) the utility of Kolmogorov’s extension theorem is due to the interpretation (K₂).

We think that this view is the most essential in all statements concerning Kolmogorov’s extension theorem.

Consider a Borel measurable space (X, \mathcal{B}_X) , where X is a (locally compact) separable complete metric space. Let Ω be a locally compact Hausdorff space with a suitable measure ν . Let g be a quantity, that is, a

continuous map $g : \Omega \rightarrow X$. Define the observable $O_g = (X, \mathcal{B}_X, G)$ in $L^\infty(\Omega, \nu)$ such that

$$[G(\Xi)](\omega) = \chi_{g^{-1}(\Xi)}(\omega) = \begin{cases} 1 & \text{if } \omega \in g^{-1}(\Xi) \\ 0 & \text{if } \omega \notin g^{-1}(\Xi) \end{cases} \quad (\forall \Xi \in \mathcal{B}_X, \omega \in \Omega). \quad (10)$$

Lemma 1 [The measurement of a quantity]. *Let $O_g = (X, \mathcal{B}_X, G)$ be the observable induced by a quantity $g : \Omega \rightarrow X$ as in (10). Let $x \in X$ be a measured value obtained by the measurement $M_{L^\infty(\Omega, \nu)}(O_g, S_{[\delta_{\omega_0}]})$. Then, we can surely believe that $x = g(\omega_0)$. That is, for any open set $D \subseteq X$ such that $g(\omega_0) \in D$, the probability that a measured value obtained by $M_{L^\infty(\Omega, \nu)}(O_g, S_{[\delta_{\omega_0}]})$ belongs to D is equal to 1.*

Proof. Let $D \subseteq \mathcal{B}_X$ be any open set such that $g(\omega_0) \in D$. According to Axiom 1, the probability that a measured value x obtained by the measurement $M_{L^\infty(\Omega, \nu)}(O_g, S_{[\delta_{\omega_0}]})$ belongs to D is given by $\chi_{g^{-1}(D)}(\omega_0) = 1$. Since D is arbitrary, we can surely believe that $x = g(\omega_0)$. \square

Lemma 2. *Consider a finite tree $(T := \{t_0, t_1, \dots, t_n\}, \leq)$ with the root t_0 . Let $\{\Phi_{t', t} : L^\infty(\Omega_{t'}) \rightarrow L^\infty(\Omega_t)\}_{(t', t) \in T_{\leq}^2}$ be a deterministic Markov relation, which is also represented by the deterministic maps $\{\phi_{\pi(t), t} : \Omega_{\pi(t)} \rightarrow \Omega_t\}_{T \setminus \{t_0\}}$. For each $t \in T$, consider a continuous map $g_t : \Omega_t \rightarrow X_t$ and consider an observable $O_{g_t} := (X_t, \mathcal{B}_{X_t}, G_t)$ in the $L^\infty(\Omega_t)$ induced by a quantity g_t . Let \widehat{O}_T be the realization of the deterministic sequential observable $\mathbb{O}_T = [\{O_{g_t}\}_{t \in T}, \{\phi_{\pi(t), t} : \Omega_{\pi(t)} \rightarrow \Omega_t\}_{T \setminus \{t_0\}}]$. Let $(x_t)_{t \in T} \in \times_{t \in T} X_t$ be a measured value obtained by the measurement $M_{L^\infty(\Omega, \nu)}(\widehat{O}_T, S_{[\delta_{\omega_0}]})$. Then, we can surely believe that $x_t = g_t(\phi_{0, t}(\omega_0))$ ($\forall t \in T$). That is, for any open set $D_t \subseteq X_t$ such that $g_t(\phi_{0, t}(\omega_0)) \in D_t$ ($\forall t \in T$), the probability that a measured value $(x_t)_{t \in T}$ obtained by $M_{L^\infty(\Omega_{t_0}, \nu_{t_0})}(\widehat{O}_T, S_{[\delta_{\omega_0}]})$ belongs to $\times_{t \in T} D_t$ is equal to 1.*

Proof. This is a slight generalization of Lemma 1. Thus, the proof is easy as follows. For each $t \in T$, consider any open set $D_t \subseteq \mathcal{B}_{X_t}$ such that $g_t(\phi_{0, t}(\omega_0)) \in D_t$ ($= (g_t \circ \phi_{0, t})(\omega_0) \in D_t$). According to Axiom 1, the probability that a measured value $(x_t)_{t \in T}$ obtained by the measurement $M_{L^\infty(\Omega_{t_0}, \nu_{t_0})}(\widehat{O}_T, S_{[\delta_{\omega_0}]})$ belongs to $\times_{t \in T} D_t$ is given by $\prod_{t \in T} \chi_{(g_t \circ \phi_{0, t})^{-1}(D_t)}(\omega_0) = 1$. This completes the proof. \square

4. Zeno's paradox in measurement theory

4.1. What is Zeno's paradox?

Now we can review the question: "What is Zeno's paradox?" in Section 1.4 as follows:

- (P₁) From **Figure 1**, choose the proper world view on which the formula (1) should be based!
- (P₂) Or, if there is no proper world view in this figure, propose and add a new world view to **Figure 1**.

Of course, we shall execute the our assertion (D) in the following section.

4.2. Zeno's paradox in measurement theory

According to the space-time problem: (M₂), define the time axis by $\widehat{T} = [0, \infty)$ with the usual order \leq . For each $t \in \widehat{T}$, consider a classical basic structure $[C_0(\Omega_t), L^\infty(\Omega_t, \nu_t)]_{B(H)}$. Here it may be usual to assume that $\Omega_t = \Omega$ ($\forall t$). Further, consider a quantity $g_t : \Omega_t \rightarrow X_t (\equiv \mathbb{R}^2)$, which, by (10), induces the observable $O_{g_t} = (X_t, \mathcal{B}_{X_t}, G_t)$ in $L^\infty(\Omega_t)$, and consider the deterministic sequential observable $\mathbb{O}_{\widehat{T}} = [\{O_{g_t}\}_{t \in \widehat{T}}, \{\phi_{t', t} : \Omega_{t'} \rightarrow \Omega_t\}_{(t', t) \in \widehat{T}_{\leq}^2}]$. Here, for any finite $T \in \mathcal{P}_0(\widehat{T})$, we have a deterministic sequential observable $\mathbb{O}_T = [\{O_{g_t}\}_{t \in T}, \{\phi_{t', t} : \Omega_{t'} \rightarrow \Omega_t\}_{(t', t) \in T_{\leq}^2}]$, which has the realization \widehat{O}_T in $L^\infty(\Omega_0)$ (cf. Lemma 2). Consider the family $\{\widehat{O}_T \mid T \in \mathcal{P}_0(\widehat{T})\}$, which clearly satisfies the consistency condition (N). Thus, by Theorem 1, we get the realization $\widehat{O}_{\widehat{T}}$ in $L^\infty(\Omega_0)$ of the sequential observable $\mathbb{O}_{\widehat{T}}$. Let $(x_t)_{t \in \widehat{T}} \in \times_{t \in \widehat{T}} X_t$ be a measured value obtained by the measurement $M_{L^\infty(\Omega_0, \nu_0)}(\widehat{O}_{\widehat{T}}, S_{[\delta_{\omega_0}]})$. Then, we can surely believe that

$$x_t = g_t(\phi_{0, t}(\omega_0)) \quad (\forall t \in \widehat{T}) \quad (11)$$

That is, for any $T \in \mathcal{P}_0(\widehat{T})$ and any open set $D_t \subseteq X$ such that $g_t(\phi_{0, t}(\omega_0)) \in D_t$ ($\forall t \in T$), the probability that a measured value $(x_t)_{t \in \widehat{T}}$ obtained by $M_{L^\infty(\Omega_{t_0}, \nu_{t_0})}(\widehat{O}_{\widehat{T}}, S_{[\delta_{\omega_0}]})$ belongs to $P_{T, \widehat{T}}^{-1}(\times_{t \in T} D_t)$ is equal to 1.

Recall the formula (1). If we put

$$(Q) \quad x_t = (\xi(t), \eta(t)) = (vt, \gamma vt + a) \quad (\forall t \in [0, \infty)),$$

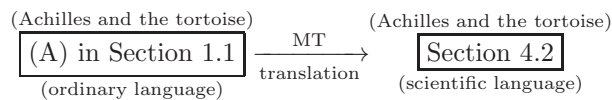
we can describe the formula (1) in terms of measurement theory. That is, in measurement theory, the formula (1) should be understood as the equation of measured values. Of course, the calculation of the (Q) is the same as that of Section 1.3.

Remark 1. (i): In [7], the above was discussed in the special case that $\Omega_t = X_t = \mathbb{R}^2$. Now we think that it is not sufficient and it should be improved as mentioned in this section. However, in the sense mentioned in the abstract, we believe that Zeno's paradoxes were already solved in measurement theory [3–13].

(ii): The beginners of philosophy may misunderstand “Achilles and the tortoise” as the elementary mathematical problem concerning infinite series. In order to avoid the confusion, we choose “Achilles and the tortoise” and not “flying arrow paradox”, though the latter is the most excellent in all Zeno's paradoxes.

5. Conclusions

What we executed in this paper is merely the translation from “ordinary language” to “scientific language”, that is,



We believe that this translation is just “the mechanical world view” or “the method of science” (at least, science in the series ① of **Figure 1**). That is, ordinary science (at least, its basic statements) should be described in terms of measurement theory. For example, for the translation of equilibrium statistical mechanics and the Monty-Hall problem, see [12] and [13] respectively.

Since Zeno's paradoxes have the long history of 2500 years, we should refrain from the immediate conclusion. However, we believe that our view (C) is the central subject of Zeno's paradoxes.

We hope that some readers will propose another powerful scientific language (as mentioned in (P₂)), and also, our assertion will be examined from the various points of view.

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